

SIR JAMES JEANS AND THE STABILITY OF GASEOUS STARS

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In 1925 Sir James Jeans calculated that a star made up of an ideal gas, generating energy as a moderately positive function of temperature and density, could not exist. Such stars would be unstable to radial oscillations of increasing size. It appears that the flaw in his calculation has never been clearly explained, especially the physical basis for it. I conclude it lies in an almost offhand assumption made about the form of the temperature perturbation. The episode provides a number of lessons about complicated calculations and their interpretation.

What is a Star? The view in 1925

In the first decades of the twentieth century one of the areas of great astrophysical interest and activity was stellar structure. It was fairly clear that the Kelvin-Helmholtz theory of gravitational contraction (which lives on in our terminology of ‘early-type’ and ‘late-type’ stars) was inadequate, though its replacement was not immediately available. The theory of relativity promised in principle a sufficiently long-lasting power source through the conversion of mass to energy, but details remained obscure. Using the nineteenth-century science of thermodynamics and the twentieth-century theories of radiation, models of gaseous stars were constructed and elaborated, eventually forming the basis of the science of stellar structure as we know it today.

But of course it was not clear at the time that such an approach would necessarily work. In fact Sir James Jeans, a major figure in astrophysics, put forward a theory involving liquid stars powered by a form of nuclear fission. A complete treatment of his theory and its fate is, however, beyond the scope of a short paper like this one. Here I intend to treat just one aspect of Jeans’ work: his calculation that gaseous stars were unstable.

With the benefit of hindsight we can say that Jeans’ calculation must have been in error. There are several possible explanations, from simple mistakes in algebra to new physics that was simply unknown in 1925. My initial aim in this study is to determine where Jeans’ error lies, and especially whether it could have been detected using methods and knowledge available to him. Subsequently I will examine some of the implications of the episode for the practice of mathematical model-building and for the process of science.

It is possible to make many more connexions between Jeans' ideas and subsequent work, but for limitations of space I will maintain the focus on his calculation and things having a direct bearing on it.

Jeans' Analysis

Jeans¹ presented his results in a *Monthly Notices* paper in 1925, but I will use as a reference the revised version² published in book form a few years later. The relevant parts are sections 105-111, found on pages 117 through 125. I will outline them here, referring the reader to the original for the details.

Jeans begins by writing the equation of motion for a shell at radius r within a star, supported by pressure and held together by gravity:

$$\frac{d^2 r}{dt^2} = -\frac{1}{\rho} \frac{d}{dr} \left(p_G + \frac{1}{3} a T^4 \right) - \frac{\gamma M_r}{r^2} \quad (1)$$

where ρ is the mass density, p_G the gas pressure, a the radiation constant, T the temperature, γ Newton's constant and M_r the mass within the radius r . Next Jeans accounts for energy raising the temperature of the material, doing pressure-volume work, being generated by some process as yet unspecified, and flowing out and in:

$$\rho C_v \frac{dT}{dt} - \left(p_G + \frac{4}{3} a T^4 \right) \frac{1}{\rho} \frac{d\rho}{dt} = \rho G - \frac{1}{r^2} \frac{d}{dr} (r^2 H) \quad (2)$$

where C_v is the heat capacity of the material, G the rate of generation of energy per unit mass, and H the radiant energy flux.

Next the model is made more specific by assuming something like an ideal gas

$$p_G \propto T \rho^{1+s} \quad (3)$$

where s may be used to parametrize departures from ideal gas behaviour; and an opacity similar to that of Kramers' expression

$$k = \frac{c \mu \rho}{T^{3+n}} \quad (4)$$

with k the coefficient of opacity, μ the molecular weight of the material and c the speed of light. If $n = 1/2$ a Kramers opacity is recovered. Assuming an Eddington grey atmosphere, this expression allows us to write the equilibrium radiation flux H in a useful form.

Jeans assumed an energy generation law in which

$$G \propto \rho^\alpha T^\beta \quad (5)$$

and, for convenience in notation, introduced λ as the ratio of gas pressure to radiation pressure.

To this point Jeans has built a star almost as one would do today. It is of course much simpler than the models one builds nowadays on a computer, but there should be nothing inherently unstable about these simplifications.

The next step is the stability analysis. The general problem of stability for a system as complicated as this is difficult and complicated to treat analytically. Through various relations and assumptions Jeans reduced the problem to that of the perturbation in size, δr ; and assumed a particular form of perturbation, a proportional change in radius (so that $\delta r/r_0$ is constant with radius). This still left too many terms in $\delta T/T_0$, so Jeans assumed also that it was constant with radius; that is, there was a proportional heating or cooling everywhere. I think this is a very important assumption, though it does not seem to have excited any comment before now.

With these assumptions and simplifications made, Jeans obtained the master equation of stability:

$$\begin{aligned} \frac{d^3}{dt^3}\delta r &+ \frac{(7+n-\beta)G_0}{C_v T_0} \frac{d^2}{dt^2}\delta r \\ &+ \frac{\gamma M_r}{r_0^3} \left[\frac{\lambda+4}{\lambda+1} \left(\frac{3p_G + 4aT_0^4}{\rho C_v T_0} - 1 \right) + \frac{3s\lambda}{\lambda+1} \right] \frac{d}{dt}\delta r \\ &+ \frac{\gamma M_r}{r_0^3} \frac{G_0}{C_v T_0} \left[\frac{\lambda+4}{\lambda+1} (3\alpha + \beta - n) + \frac{3s\lambda}{\lambda+1} (7+n-\beta) \right] \delta r = 0. \end{aligned} \quad (6)$$

If one cares to reproduce the algebra, one finds also that Jeans has assumed that λ does not vary with radius, nor C_v . In any realistic star they will; but not so much as to change the qualitative stability.

Equation (6) is a linear equation with constant coefficients (constant with respect to time; in general they may vary from shell to shell) so the solutions will be of the form e^{wt} . There are three possible values for w , solutions of the equation

$$w^3 + Bw^2 + Cw + D = 0 \quad (7)$$

where B is the coefficient of $d^2\delta r/dt^2$ in Equation (6), and similarly. Following Jeans we note that any positive w , or complex w with a positive real part, denotes instability. Avoiding a real, positive w gives the condition that D be positive. If we assume a perfect gas, so $s = 0$, this means

$$3\alpha + \beta - n > 0. \quad (8)$$

So if the coefficients of energy generation are too small, the star will monotonically shrink or expand. Physically, this means that if compressing the star (say) does not generate enough additional heat to cause the pressure to rise and bounce back, the star keeps on shrinking. This is reasonable, and so far we have no argument with Jeans' calculations.

Avoiding a complex w with a positive real part is a bit more involved, but if D is positive the condition reduces to

$$BC > D \quad (9)$$

which means (for $s = 0$)

$$(7+n-\beta) \left(\frac{3p_G + 4aT_0^4}{\rho C_v T_0} - 1 \right) > 3\alpha + \beta - n. \quad (10)$$

(The expression Jeans gives is a bit different, but in the last step of his derivation he appears to have made a substitution that is true only for critical stability, that is, only when the inequality is an equation. For purposes of finding the critically stable exponents no trouble should have resulted, however.)

Taking Equation (10) together with Equation (8), we find that the fraction in the brackets (something like the ratio of pressure-energy to thermal energy) must be greater than unity. For a star in which only gas pressure is important it has the value $3/2$, while if radiation pressure dominates it approaches 1. At unity, the two stability criteria together require

$$3\alpha + \beta - n = 0 \quad (11)$$

an extremely restrictive condition; if a two-particle fusion reaction is postulated as an energy source so that $\alpha = 1$, along with a Kramers opacity, it must go *slower* as temperature increases, and at an exact rate. The situation with a ratio of $3/2$ is better, but still a two-particle reaction cannot depend even linearly on the temperature. Jeans' stability criteria exclude essentially all nuclear fusion reactions as a possible source of stellar energy.

Oscillations and the Thermal Instability

What is the *physical* cause of Jeans' unstable stars? He interpreted the oscillations of increasing amplitude as being due to the increased heat-energy liberated by the reaction (whatever it is) during the dense phase of the oscillation requiring an expansion of a larger amplitude during the following phase. This is a plausible interpretation, but worth looking at in more detail.

The perturbed energy equation (from Equation 2) is

$$\begin{aligned} \rho C_v T \frac{d}{dt} \left(\frac{\delta T}{T} \right) &- \rho G (\beta - 7 - n) \frac{\delta T}{T} = \\ &- 3 \left(C_v T \rho^{1+s} + \frac{4}{3} a T^4 \right) \frac{d}{dt} \left(\frac{\delta r}{r} \right) - \rho G (3\alpha + 7) \frac{\delta r}{r} \end{aligned} \quad (12)$$

where the subscript noughts, meaning equilibrium values, have been suppressed for ease of notation. Now suppose we impose a sinusoidal oscillation in $\delta r/r$. The equation may now be thought of as a linear, first-order differential equation in the (proportional) temperature perturbation with a sinusoidal forcing function. The solution will be the sum of sinusoidal terms and the homogeneous solution, which is

$$\frac{\delta T}{T} \propto \exp \left(\frac{G}{C_v T} (\beta - 7 - n) t \right). \quad (13)$$

In order for the temperature to stay bounded, $\beta \leq 7 + n$. This is not as strict a requirement as Jeans found, but then this is a *different* calculation: we have found the temperature response to a forced radial motion of the star. It still puts an upper limit on the temperature sensitivity of the reaction, and in particular excludes the exponential dependence one expects for nuclear fusion reactions. Jeans' interpretation is thus shown to be correct. In addition, having a physical source for Jeans' instability allows us to investigate more complicated situations.

Previous Identifications of the Flaw

For a calculation that appeared to show gaseous, fusion-powered stars impossible, Jeans' work has left very little lasting trace. Chandrasekhar's³ 1939 work on stellar structure, very extensive in its listing of the literature, does not even mention it directly. He does refer to an unspecified 'belief' in instability resulting from a certain kind of energy-producing reaction (pp. 457, 468), but sees no convincing reason for it. Milne's⁴ biography of Jeans describes the calculation in some detail, but as far as criticism notes only (p. 138) that in the calculation he parametrises departures from the ideal gas laws in two incompatible ways. This does show that Jeans was not terribly interested in specifically how stars might depart from being ideal gases, but then Jeans says just that himself at the beginning of his calculation. At any rate, it has no effect on the stability analysis for ideal-gas stars. (It is possible that Milne thought Jeans' work might be essentially correct; in his description he shows scepticism about the gaseous model.) The much more recent work on pulsating stars by Cox⁵ does mention Jeans' calculation (pp. 166 and 172). It is noted as being equivalent to a one-zone model of a star (an important point, and one discussed below), but no analysis of its flaw is given.

What appears to be the accepted answer for the flaw in Jeans' calculation appears in a pair of papers by Cowling^{6,7} (the first of which is cited by Chandrasekhar). He pointed out that a proportional change in radius, with $\delta r/r$ constant in space, probably was not a normal mode for a star. Therefore (putting words in his mouth) giving it such a kick would excite a number of normal modes of different frequencies, and the perturbation would not remain a constant proportion. Jeans' perturbation calculation was then not strictly a *stability* determination, in which the star is left to itself after being kicked, but shows a response to a specific kind of forcing.

Cowling had found a flaw in Jeans' work that made its result questionable. But this is not the same thing as showing that the flaw in fact led to the erroneous conclusion. Indeed, it is not immediately clear how a combination of normal motions, each of them individually stable, should be unstable taken together. (More recently, Papaloizou^{8,9} has shown that at least for some models of very high-mass stars, the opposite can happen: mode mixing can stabilise otherwise unstable modes.)

Cowling did go on to impose a proportional radial oscillation on his analysis, and obtained essentially Jeans' result. But because he was dealing with oscillations assumed to be adiabatic or nearly so, the temperature perturbation was automatically of the same form as the radial perturbation; so Jeans' separate assumption about it was implicitly included. It is this assumption concerning the temperature perturbation that I think is the real problem with Jeans' analysis. The next step is to test this idea.

Non-Proportional Temperature Perturbations

The straightforward way to check the effect of this assumption is to relax it. One perturbs Equations (1) and (2) again, this time allowing terms with various

derivatives of $(\delta T/T)$ with radius, then collecting everything into a new and more accurate version of Equation (6). In principle it is possible. But instead of requiring one additional equation, the time derivative of Equation (1), and a bit of algebra, one needs nine additional equations and a truly horrible lot of algebra. It is not a practical pen-and-paper exercise and I do not believe Jeans would have seriously considered it.

Even writing down Equation (12) as it appears with the relaxation of the constraint on the temperature perturbation is not particularly enlightening. But by making the further assumption that the temperature perturbation is separable, so that

$$\frac{\delta T}{T} = R(r)X(t). \quad (14)$$

the expanded form of Equation (12) can be rearranged into

$$\begin{aligned} & \rho C_v T R X' + \\ & \left[-\rho G (\beta - 7 - n) R - \left((7 + n)H + \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2 T H}{dT/dr} \right) \right) R' - \frac{HT}{dT/dr} R'' \right] X \\ & = -3 \left(C_v T \rho^{1+s} + \frac{4}{3} a T^4 \right) \frac{d}{dt} \left(\frac{\delta r}{r} \right) - \rho G (3\alpha + 7) \frac{\delta r}{r} \end{aligned} \quad (15)$$

where primes denote derivatives with respect to the independent variable. If we impose a sinusoidal variation in the radial perturbation as before and consider this an equation for X , we again have a first-order differential equation with a forcing term. This time the coefficients depend on the variation of the temperature perturbation with radius, which we do not know. But some qualitative reasoning about the relative sizes of the R , R' and R'' coefficients shows that the radial change in temperature perturbation need only be a small fraction of the perturbation itself to dominate the stability calculation.

Physically, Jeans' assumption of a strictly proportional temperature perturbation has the effect of pumping heat energy around the star in an unphysical way. It is not a large effect, taking many oscillations to grow significantly (so that the adiabatic assumption of Cowling and many others is a very good one for working out small pulsations), but enough to blow up the star eventually.

On a more abstract level, it appears that the problem with Jeans' analysis lies in ignoring the internal structure of the star, in requiring it to be in some sense a single unit. We may explore this idea with a toy star.

The Toy Star

Not as a serious mathematical model, but as a way of investigating some of the effects we expect to operate in a real star, I present here a very simple model. (This is similar in motivation and in general to that in Kippenhahn and Weigert⁸, pp. 13-15, 235-8 and 407-8, though the details and the application are different.) Suppose we enclose a spherical quantity of ideal gas, of mass m and density ρ , with some mixture of radiation inside a hollow shell of radius r and mass M . The gas mass m is much smaller than M ; the shell is supported

by the pressure of gas and radiation, and is held in by gravity. Everything is at one temperature T and the surface radiates heat as a blackbody. Inside the shell there is an energy-generating reaction that goes as $\rho^\alpha T^\beta$. The equations of motion and energy balance are thus

$$\begin{aligned} M \frac{d^2 r}{dt^2} &= -\frac{\gamma M^2}{r^2} + \left(p_G + \frac{1}{3} a T^4 \right) 4\pi r^2 \\ mG &= mC_v \frac{dT}{dt} + \left(p_G + \frac{4}{3} a T^4 \right) 4\pi r^2 \frac{dr}{dt} + 4\pi r^2 \sigma T^4. \end{aligned} \quad (16)$$

Perturbing these, the heat balance equation becomes

$$U \frac{d}{dt} \left(\frac{\delta T}{T} \right) = (\beta - 4) F_0 \frac{\delta T}{T} - (\lambda + 4) \frac{4r}{3c} F_0 \frac{d}{dt} \left(\frac{\delta r}{r} \right) - (3\alpha + 2) F_0 \frac{\delta r}{r} \quad (17)$$

where $U = \rho C_v T$ is the (unperturbed) thermal heat content and $F_0 = 4\pi r^2 \sigma T^4$ is the (unperturbed) energy radiated away from the surface. The master stability equation is derived as before. The first condition for stability, avoiding a monotonic expansion or collapse, implies

$$3\alpha + \beta - 2 > 0 \quad (18)$$

a lower limit on the energy-generating reaction. The second criterion reduces to

$$(4 - \beta) (\lambda + 4) \frac{4r F_0}{3cU} > 3\alpha + 2. \quad (19)$$

The fraction can be interpreted as F_0/U , the reciprocal of the time required to radiate away internal heat energy without replacement; times $4/3$ the time required for light to go from the centre of the sphere to the surface. For even a toy star the ratio of crossing time to radiative timescale should be very, very small, giving us almost independently of the value of β ,

$$\alpha < -\frac{2}{3} \quad (20)$$

that is, that the energy-generating reaction is required to go *more slowly* with increasing density.

If we look again at Equation (17) and impose a sinusoidal radial oscillation, we find that the temperature perturbation response is given by a sinusoidal term plus

$$\frac{\delta T}{T} \propto e^{(\beta-4) \frac{F_0}{U} t} \quad (21)$$

that is, if the energy generation reaction produces energy faster than it can be radiated away, temperature increases exponentially on a radiative timescale.

Taken together, these results show the toy star behaving in a similar way to Jeans' gaseous stars, and for similar reasons. The energy generation rate must be sensitive to temperature and pressure, but in an extremely restrictive

and unphysical way. Too little sensitivity and the star implodes or explodes monotonically; too much and growing oscillations, due to excess heat energy, tear the star apart.

The Stability of Gaseous, Fusion-Powered Stars

It is important to keep in mind just what all these calculations have, and have not, proven. Cowling called attention to the fact that (in my words) Jeans' analysis was not strictly a stability calculation, since the required form of perturbation was not a free one. Since he recovered Jeans' result by imposing Jeans' restrictive form of the radial perturbation, he concluded that this was the problem. It is true enough for most purposes, since pulsations are very nearly adiabatic and for those the perturbations are proportional. By focussing on the temperature perturbation I have shown that Jeans' physical explanation for his instability is correct, that is, that each oscillation (of the restricted type) produces more heat energy than steady motion dissipates. I have then shown that even a small departure from a strictly proportional temperature perturbation could plausibly stabilise a forced oscillation, so the physical mechanism that destabilises a Jeans star does not operate. Thus a full stability analysis using his methods (possible in principle, though not likely in practice) would show that gaseous, fusion-powered stars are stable. Since Jeans' assumptions of the form of temperature and radius perturbations in some sense ignore the structure of the star, I constructed a structureless toy star, and found it to be subject to the same kind of instability that Jeans found, thereby lending support to that interpretation.

In all this I have *not* proven the stability of gaseous, fusion-powered stars. Indeed it seems almost certain that such proof is beyond the specific techniques used by Jeans. Cowling^{6,7} took a more in-depth approach, depending more on the details of gaseous stars (some of which had not been determined when Jeans wrote). In that sense he was less general; but in allowing any shape to the perturbations he was more general in the more important way. In the end, his work was accepted as the true answer.

It appears that the problem was inherently more complicated than Jeans allowed for, not allowing of reduction to a single ordinary differential equation. It was, however, possible for him to see that relaxing the assumption concerning the temperature perturbation *could* have changed the conclusion, and thus that the calculation was actually inconclusive.

Lessons for Mathematical Modelling and the Progress of Science

Out of all this algebra has come one potentially useful insight into the structure of stars: a star *must* change its structure to be stable. Simple proportional expansion and cooling, or contraction and heating, is not quite enough. Our concern, however, is much more with the implications of the episode for the practice of science.

Jeans' mistaken calculation had very little direct effect on the progress of stellar modelling. Its main importance for us here lies, first, in its implications

for the continuing practice of constructing mathematical models of astrophysical systems. The actual flaw was subtle in its introduction and effect and went undetected for a very long time. The lesson that mistakes happen and that they may not be found very soon is not a new one, but well worth underlining. Also worth emphasising is the fact that every mathematical assumption has some physical implication and that the true influence of a simplifying assumption is not known until it is relaxed. Models have grown no simpler since Jeans' day, but simplifying assumptions with all their effects must still be made.

This episode is important, secondly, for its implications about the progress of science, *exactly* because it had little effect. A calculation that appeared to disprove a popular picture (it hardly yet amounted to a theory) made little impact, in spite of the fact that its flaw was not discovered for a decade. Indeed, one may say the flaw was never actually found, but rather a different (more complicated and more correct) calculation was performed that replaced Jeans' effort. Note that Cowling was investigating in great detail the stability of *gaseous* stars, not the liquid versions postulated by Jeans. This episode, then, illustrates the process of accepting a new theory as noted in Kuhn's¹¹ original work (notably chapter XII): the new theory need not answer all questions at once, and indeed might not even do as well as another theory in specific places. A theory is almost necessarily a vague and imperfect thing at the beginning. It is only after a long period of refinement that its true power is shown.

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